A Level Mathematics B (MEI)

H640/03 MEI Pure Mathematics and Comprehension

Question Set 1

1 Triangle ABC is shown in Fig. 1.

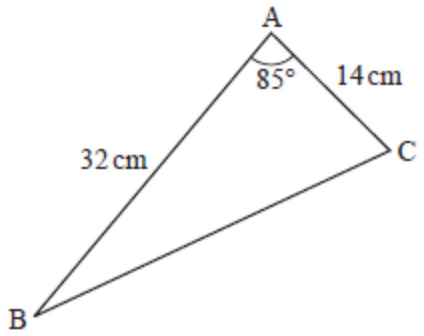


Fig. 1

Find the perimeter of triangle ABC. **[3]**

Using the cosine rule : $BC^2 = 32^2 + 14^2 - 2 \times 32 \times 14 \times \cos 85$
$= 1142$ $BC = 33.8$

$33.8 + 32 + 14 = 79.8$

2 The curve $y = x^3 - 2x$ is translated by the vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Write down the equation of the translated curve. **[2]**

$y = (x+1)^3 - 2(x+1) - 4$
$y = x^3 + 3x^2 + x - 5$

3 Fig. 3 shows a circle with centre O and radius 1 unit. Points A and B lie on the circle with angle AOB $= \theta$ radians. C lies on AO, and BC is perpendicular to AO.

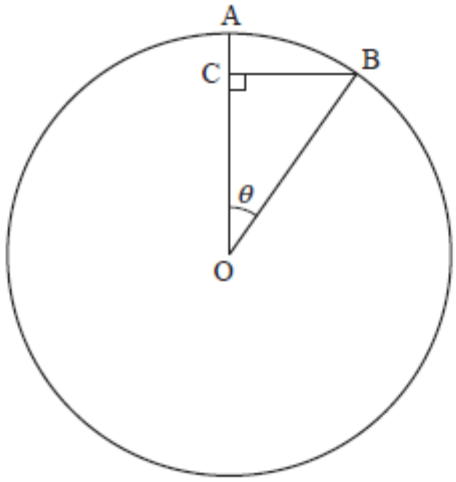


Fig. 3

Show that, when θ is small, AC $\approx \frac{1}{2}\theta^2$. **[2]**

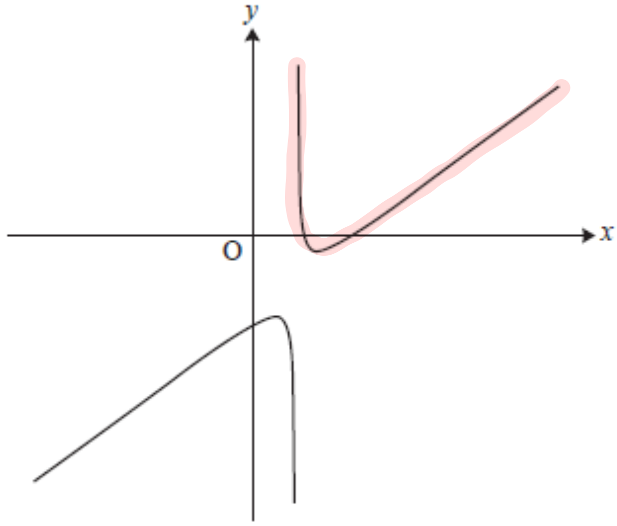
$OC = 1\cos\theta = \cos\theta$

when θ is small $= 1 - \frac{\theta^2}{2}$

$AC = OA - OC = 1 - (1 - \frac{\theta^2}{2}) = \frac{\theta^2}{2}$

4 **In this question you must show detailed reasoning.**

A curve has equation $y = x - 5 + \frac{1}{x-2}$. The curve is shown in Fig. 4.



(a) Determine the coordinates of the stationary points on the curve. **[5]**

$y = x - 5 + (x-2)^{-1}$

$\frac{dy}{dx} = 1 - (x-2)^{-2}$

stationary point $\frac{dy}{dx} = 0$

$0 = 1 - (x-2)^{-2}$

$\frac{1}{(x-2)^2} = 1 \rightarrow 1 = (x-2)^2$

$0 = (x-2)^2 - 1$

$0 = x^2 - 4x + 4 - 1$

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x = 3$ OR $x = 1$

$y = 3 - 5 + \frac{1}{3-2}$
$= 3 - 5 + 1$
$= -1$

$y = 1 - 5 + \frac{1}{1-2}$
$= 1 - 5 - 1$
$= -5$

$\therefore$ Stationary points are $(3, -1)$ or $(1, -5)$

(b) Determine the nature of each stationary point. **[3]**

from diagram $(3, -1)$ = minima.

$\frac{d^2y}{dx^2} = 2(x-2)^{-3} \rightarrow x = 3, \; 2(3-2)^{-3} = 2 \cdot 1^{-3} = \frac{2}{1} = 2$

$2 > 0 \;\therefore$ minima

from diagram $(1, -5)$ = maxima

$\frac{d^2y}{dx^2} = 2(1-2)^{-3} = 2(-1)^{-3} = \frac{2}{-1^3} = -2$

$-2 < 0 \;\therefore$ maxima

(c) Write down the equation of the vertical asymptote. $x = 2$ **[1]**

(d) Deduce the set of values of x for which the curve is concave upwards. **[1]**

$x > 2$

5 A social media website launched on 1 January 2017. The owners of the website report the number of users the site has at the start of each month. They believe that the relationship between the number of users, n, and the number of months after launch, t, can be modelled by $n = a \times 2^{kt}$ where a and k are constants.

(a) Show that, according to the model, the graph of $\log_{10} n$ against t is a straight line. **[2]**

$\log_{10} n = \log_{10}(a \times 2^{kt}) \longrightarrow \log_{10}(a \times 2^{kt}) = \log_{10} a + kt \log_{10} 2$

$\therefore \log_{10} n = \log_{10} a + kt \log_{10} 2$

= it is a straight line, gradient = $k\log_{10} 2$ which is constant

(b) Fig. 5 shows a plot of the values of t and $\log_{10} n$ for the first seven months. The point at $t = 1$ is for 1 February 2017, and so on.

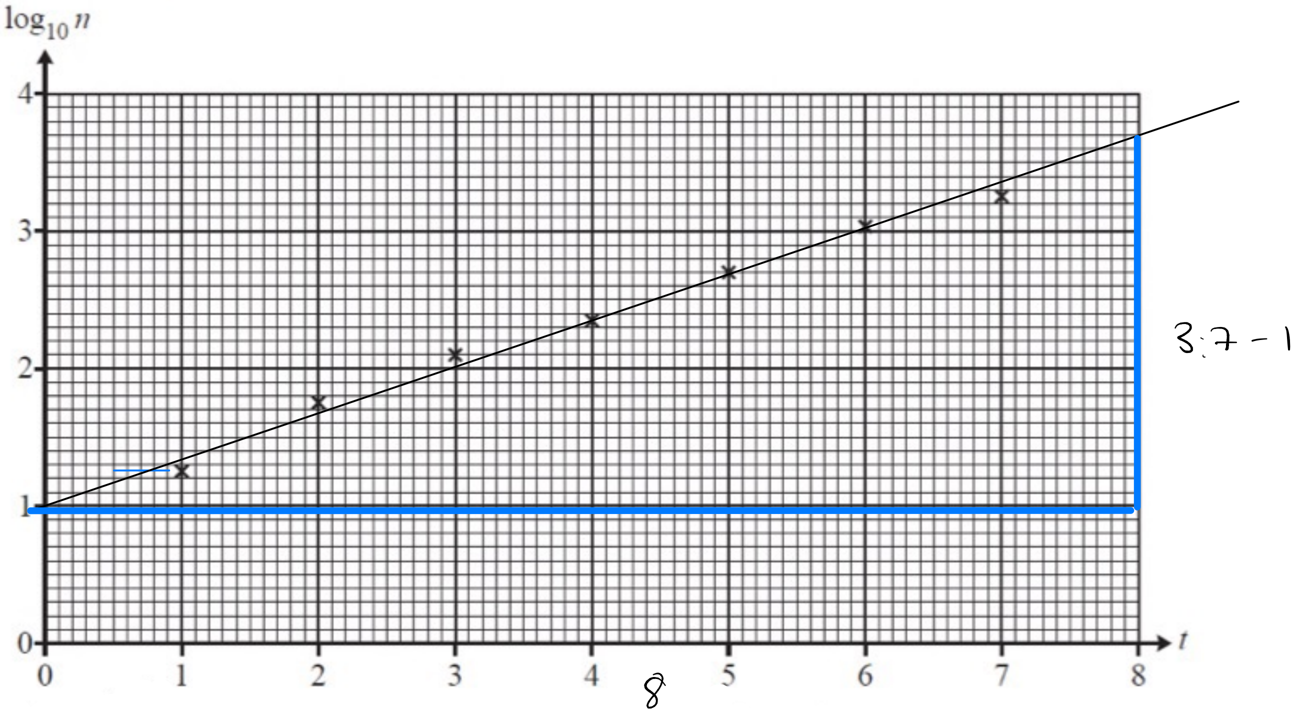


Fig. 5

Find estimates of the values of a and k. **[4]**

$\log_{10} n = 1$ (y-intercept)

$\therefore \log_{10} a = 1$

$a = 10^1 = 10$

gradient $= \dfrac{3.7 - 1}{8 - 0} = 0.3375$

$k \log_{10} 2 = 0.3375$

$k = \dfrac{0.3375}{\log_{10} 2} = 1.12115..$

$k = 1.12$ (3sf)

$a = 10$

(c) The owners of the website wanted to know the date on which they would report that the website had half a million users. Use the model to estimate this date. **[4]**

$500\,000 = 10 \times 2^{1.12115t}$

$1.12115t \log 2 = \log 50000$

$t = \dfrac{\log 50000}{1.12115 \log 2} = 13.92287409 \approx 14$ months

$t = 1/3/18$

(d) Give a reason why the model may not be appropriate for large values of t [1]

The popularity will reach a max

6 Find the constant term in the expansion of $\left(x^2+\frac{1}{x}\right)^{15}$ [2]

$^{15}C_5 (x^2)^5 \left(\frac{1}{x}\right)^{10}$

$^{15}C_5 \left(\frac{x^{10}}{x^{10}}\right) = {}^{15}C_5 = 3003$

7 **In this question you must show detailed reasoning.**

Fig. 7 shows the curve $y = 5x - x^2$.

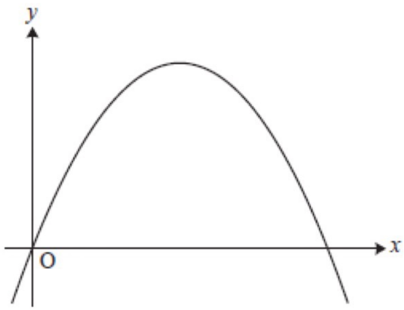


Fig. 7

The line $y = 4 - kx$ crosses the curve $y = 5x - x^2$ on the x-axis and at one other point.

Determine the coordinates of this other point. [8]

8 (a) **In this question you must show detailed reasoning.**

$y = x(5 - x)$

when $y = 0$, $x = 0$ or $x = 5$

with $y = 4 - kx$, when $x = 0$, $y = 4$ $\therefore$ can not cross ther

$\therefore$ must meet at $5, 0$

$\Rightarrow 0 = 4 - 5k$

$5k = 4$

$k = 0.8$

$\rightarrow 4 - \frac{4}{5}x = 5x - x^2$

$20 - 4x = 25x - 5x^2$

$5x^2 - 29x + 20 = 0$

$x = 5$ or $x = \frac{4}{5}$

$(5, 0)$ ✓

when $x = \frac{4}{5}$

$y = 4 - \frac{4}{5} \times \frac{4}{5}$

$y = 4 - \frac{16}{25}$

$= \frac{84}{25} = 3.36$

$\therefore$ other point is

$(4/5, 84/25)$

8 **(a)** **In this question you must show detailed reasoning.**

Determine the gradient of the curve at the point where $t = 1$. [5]

(b) Verify that the cartesian equation of the curve is $x^3 + y^3 = xy$. [3]

no curve given to answer ∴ can't answer

9

The function $f(x) = \frac{e^x}{1-e^x}$ is defined on the domain $x \in \mathbb{R}, x \neq 0$.

(a) Find $f^{-1}(x)$. [3]

$y = \frac{e^x}{1-e^x}$

$y - ye^x = e^x$

$y = e^x(1+y)$

$\frac{y}{1+y} = e^x$

$\ln \frac{y}{1+y} = x$

$\therefore f^{-1}(x) = \ln \frac{x}{1+x}$

(b) Write down the range of $f^{-1}(x)$. $f^{-1}(x) \in \mathbb{R},\ f^{-1}(x) \neq 0$ [1]

10

Point A has position vector $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ where a and b can vary, point B has position vector $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ and point C has position vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. ABC is an isosceles triangle with AC = AB.

(a) Show that $a - b + 1 = 0$. [4]

$AB = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 4-a \\ 2-b \\ 0 \end{pmatrix}$

$AC = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 2-a \\ 4-b \\ 2 \end{pmatrix}$

$\sqrt{(4-a)^2 + (2-b)^2 + 0^2} = \sqrt{(2-a)^2 + (4-b)^2 + 2^2}$

$(4-a)^2 + (2-b)^2 = (2-a)^2 + (4-b)^2 + 4$

$16 - 8a + \cancel{a^2} + 4 - 4b + \cancel{b^2} = 4 - 4a + \cancel{a^2} + 16 - 8b + \cancel{b^2} + 4$

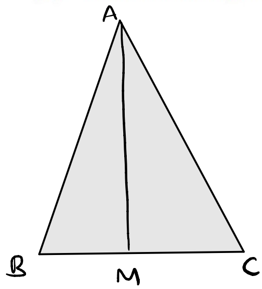
$20 - 8a - 4b = 24 - 4a - 8b$

$4b = 4a + 4$

$0 = 4a - 4b + 4$

$\underline{\underline{0 = a - b + 1}}$

(b) Determine the position vector of A such that triangle ABC has minimum area. [6]



M = midpoint of BC

$M = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

$AM = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 3-a \\ 3-b \\ 1 \end{pmatrix}$

$a+1 = b \quad \therefore \quad AM = \begin{pmatrix} 3-a \\ 2-a \\ 1 \end{pmatrix}$

$\text{Area} = \frac{1}{2} \times |\vec{AM}| \times |\vec{BC}|$

$|\vec{AM}| = \sqrt{(3-a)^2 + (2-a)^2 + 1^2}$

$= \sqrt{9 - 6a + a^2 + 4 - 4a + a^2 + 1}$

$= \sqrt{14 - 10a + 2a^2}$

$\vec{BC} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \quad \therefore \quad |\vec{BC}| = \sqrt{-2^2 + 2^2 + 2^2}$

$= 2\sqrt{3}$

$\text{Area} = \dfrac{2\sqrt{3}\sqrt{14 - 10a + 2a^2}}{2} = \sqrt{3}\sqrt{14 - 10a + 2a^2}$

$= \sqrt{3}\sqrt{2(a - 2.5)^2 + 0.75}$

$\therefore a = 2.5$ is the minimum

Position vector $= \underline{\underline{\begin{pmatrix} 2.5 \\ 3.5 \\ 0 \end{pmatrix}}} \rightarrow b = 2.5 + 1$

Resource Materials

Question Set No: 1

Fig. 5